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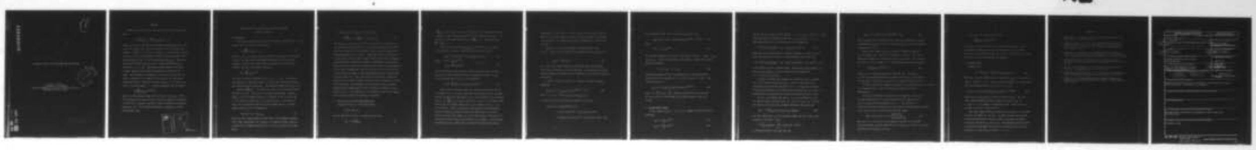

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ESTIMATING THE LAG IN A NONLINEAR TIME SERIES MODEL

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ABSTRACT

Suppose that two discrete time series $\{x_t\}$ and $\{y_t\}$ satisfy the model

$$y_t = \sum_s h_s x_{t-s} + \sum_s a_s x_{t-s-d} x_{t-s} + \varepsilon_t,$$

where $\{x_t\}$ and $\{\varepsilon_t\}$ are mutually independent Gaussian processes. The lag d, the weights $\{a_t\}$, and the spectrum of the error process are unknown. If $\sum_t a_t = 0$ or $c_x(d) = 0$, where $c_x(\tau)$ denotes the covariance function of $\{x_t\}$, then the nonlinear effect is not detectable in the spectrum $S_y(\omega)$ of $\{y_t\}$, or in the cross spectrum $S_{xy}(\omega)$. The cross bispectrum, denoted $S_{xy}(\omega_1,\omega_2)$, has a "hidden periodicity" as a function of $\kappa = \omega_1 + \omega_2$. The period is proportional to the lag d. A simple smoothed cross bispectrum estimator is defined in the paper. This estimator, denoted $B(\kappa)$, is a function of the discrete Fourier transforms of the samples of the $\{x_t\}$ and $\{y_t\}$ processes. Assume that $S_x(\omega)$ is constant in a band $(0,2\pi f_o)$. Let $\hat{d}$ denote the value of δ in the range $0 < \delta < T/2$ which maximizes the periodogram

$$T_o^{-1} \left|\sum_{k=1}^{T_o} B(\kappa_k) e^{i(\kappa_k \delta/2)}\right|^2,$$

where T_o is the largest integer such that $T_o/T \leq f_o$ and $\kappa_k = 4\pi k/T$. The periodogram has a peak of order $O(T_o)$ at $\delta=d$ against a background or order $O(1)$. A heuristic approach is used to develop the asymptotic properties of $\hat{d}$. The estimation procedure is extended to a model with multiple lags.

Estimating the Lag in a Nonlinear Time Series Model

Melvin J. Hinich

Introduction

Suppose that two discrete-time real stochastic processes $\{x_t\}$ and $\{y_t\}$ satisfy the time invariant linear model

$$y_t = \sum_{s=-\infty}^{\infty} h_s x_{t-s} + \varepsilon_t, \tag{1}$$

where $\{\varepsilon_t\}$ is a zero mean noise process that is assumed to be independent of $\{x_t\}$. In time series terminology (Brillinger [1]), $\{Ey_t\}$ is the output of a linear filter whose impulse response is $\{h_t\}$, and whose transfer function is the Fourier transform

$$H(\omega) = \sum_{t=-\infty}^{\infty} h_t e^{-i\omega t}$$

The filter is called realizable if $h_{-1} = h_{-2} = \ldots = 0$. In economics, the model with a realizable filter is called a distributed lag relationship between the two time series. The principal domain for $H(\omega)$ is $0 \le \omega < \pi$.

Assume that $\sum_t |h_t| < \infty$, and that $\{x_t\}$ and $\{\varepsilon_t\}$ are real stationary Gaussian processes. Then $\{y_t\}$ is stationary and Gaussian. Suppose, however, that the impulse response is a function of the x_t with a lag. In this case the process $\{y_t\}$ is no longer Gaussian. To be specific about the relationship between the filter weights and the x_t values, suppose that h_s in (1) is replaced by

$$h_s(\{x_t\}) = h_s + a_s x_{t-s-d} \tag{2}$$

where $\{a_s\}$ are unknown weights and the delay d is an unknown parameter. Such a model approximates the response of an amplifier whose linearity is destroyed by feedback inductances in its circuit caused by the input signal.

When $h_s(\{x_t\})$ replaces h_s in (1), we have

$$y_t = \sum_s h_s x_{t-s} + \sum_s a_s x_{t-s-d} x_{t-s} + \varepsilon_t. \tag{3}$$

The term $x_{t-s-d} x_{t-s}$ is the t-sth value of the d+1 st lag process of $\{x_t\}$ (Nelson and Van Ness [5]). If the lag d were known, the investigator can estimate as by least squares. When the lag is unknown, the estimation problem is non-trivial. Kedem [4] presents an estimator of the lag using a measure called lagged coherence, but even the asymptotic properties of his estimator are unknown. Regardless of the procedure which is used to estimate d, the analyst who is aware that the parameters of his impulse response are functions of the x_t's is better off than one who analyses the data or makes predictions based on a misspecified stationary model (1). This paper presents a method for estimating d which is easy to compute from the discrete Fourier transform of a sample of the two series. The estimator uses statistics which a frequency domain oriented analyst would use in the course of building a linear predictor of y_t. The estimate of d is a diagnostic tool for an analyst who is skeptical about the linearity of his model.

1. The Cross Spectrum and Cross Bispectrum

Assume that $Ex_t = 0$. Using the notation

$$c_x(\tau) = Ex_{t-\tau} x_t$$

for the covariance of $\{x_t\}$, it follows from (3) that

$$Ey_t = c_x(d) \sum_s a_s \tag{4}$$

If $\sum_s a_s = 0$ or $c_x(d) = 0$, then the analyst gets no information about the existence of the nonlinear component from the sample mean of $\{y_t\}$. For the rest of this paper assume that $c_s(d)\sum_s a_s = 0$. Also assume that $\sum_\tau |\tau| c_x^2(\tau) < \infty$.

Since $\{x_t\}$ is Gaussian, $Ex_{t_1}x_{t_2}x_{t_3} = 0$ for all t_1, t_2, and t_3. Thus from (3) it is easy to show that the spectrum of $\{y_t\}$ is

$$S_y(\omega) = |H(\omega)|^2 S_x(\omega) + |A(\omega)|^2 S_z(\omega) + S_\varepsilon(\omega), \tag{5}$$

where
$$A(\omega) = \sum_s a_s e^{-i\omega s}, \qquad 0 \le \omega < \pi$$
$$S_x(\omega) = \sum_\tau c_x(\tau) e^{-i\omega\tau} \tag{6}$$

is the spectrum of $\{x_t\}$, $S_\varepsilon(\omega)$ is the spectrum of the error process, and $S_z(\omega)$ is the spectrum of the d+1 st lag process $Z_t(d) = x_{t-d}x_t$. By the Gaussian nature of $\{x_t\}$, it follows that

$$S_z(\omega) = \sum_\tau [c_x^2(\tau) + c_x(\tau-d)c_x(\tau+d)] e^{-i\omega\tau} \tag{7}$$

Suppose that the analyst knows the transfer function $H(\omega)$ but not the shape of $S_\varepsilon(\omega)$. He then cannot detect the presence of the nonlinear term by estimating the spectrum of $\{y_t\}$, or the spectrum of the residuals process $\{y_t - \sum_s h_s x_{t-s}\}$, namely $|A(\omega)|^2 S_z(\omega) + S_\varepsilon(\omega)$. On the other hand, suppose that the analyst knows that S_ε is a constant for all ω, i.e. $\{\varepsilon_t\}$ is white. Then he can identify the presense of the nonlinearity from the residuals spectrum provided that $|A(\omega)|^2 S_z(\omega)$ is not flat. This will not be true, however, if $\{x_t\}$ is white and A is flat since it can be seen from (7) that S_z is flat when $\{x_t\}$ is white. In other words, an analyst who uses a white input to test a flat filter cannot detect

nonlinearity in the filter from the spectrum of the output even if he knows that the error process is white. There is usually no compelling reason, however, to assume that the noise is white even if the filter is linear.

Moving on to the cross spectrum, it follows from (3) that $Ex_{t-\tau}y_t = \Sigma_s h_s c_x(\tau-s)$, and thus the cross spectrum, denoted $S_{xy}(\omega)$, is

$$S_{xy}(\omega) = H(\omega)S_x(\omega). \tag{8}$$

As a result, the analyst obtains no information about the nonlinear term from the estimate of the cross spectrum. An estimate of the cross spectrum provides an estimate of the linear part of the transfer function, namely $H(\omega)$.

In order to study the nonlinear component, consider the <u>cross bispectrum</u>

$$S_{xy}(\omega_1,\omega_2) = \Sigma_{\tau_1}\Sigma_{\tau_2}c_{xy}(\tau_1,\tau_2)e^{-i(\omega_1\tau_1 + \omega_2\tau_2)}, \tag{9}$$

where $0 \leq \omega_1,\ \omega_2 < \pi$, and $c_{xy}(\tau_1,\tau_2) = Ex_{t-\tau_1}x_{t-\tau_2}y_t$.

It follows from (3) and the Gaussian nature of $\{x_t\}$ that

$$\begin{aligned} c_{xy}(\tau_1,\tau_2) &= \Sigma_s a_s[c_x(d)c_x(\tau_1-\tau_2) + \\ &\quad + c_x(\tau_1-s)c_x(\tau_2-s-d) + c_x(\tau_2-s)c_x(\tau_1-s-d)] \\ &= \Sigma_s a_s[c_x(\tau_1-s)c_x(\tau_2-s-d) + c_x(\tau_2-s)c_x(\tau_1-s-d)]. \end{aligned} \tag{10}$$

Thus from (8) and (9), the cross bispectrum has the form

$$S_{xy}(\omega_1,\omega_2) = A(\omega_1 + \omega_2)S_x(\omega_1)S_x(\omega_2)(e^{-i\omega_1 d} + e^{-i\omega_2 d}), \tag{11}$$

where

$$A(\omega) = \Sigma_{s=-\infty}^{\infty} a_s e^{-i\omega s} . \tag{12}$$

The principal domain for this cross bispectrum is $\{0\leq\omega_1\leq\pi,\ -\pi<\omega_2\leq\pi:\ \omega_1\geq\omega_2$ and $\omega_1\geq-\omega_2\}$. Rather than working in this domain, consider the transformation of variables

$$\kappa = \omega_1 + \omega_2, \quad \eta = \omega_1-\omega_2. \tag{13}$$

The principal domain of the cross bispectrum as a function of this variables is the triangular set $D = \{0\leq\kappa\leq2\pi,\ 0\leq\eta\leq2\pi:\ \kappa + \eta\leq2\pi\}$. The cross bispectrum is

$$S_{xy}(\kappa,\eta) = S_x(\omega_1)S_x(\omega_2)A(\kappa)\cos(\eta d/2)e^{-i(\kappa d/2)}, \tag{14}$$

where $\omega_1 = \frac{\kappa+\eta}{2}$ and $\omega_2 = \frac{\kappa-\eta}{2}$. Equation (14) will be the cornerstone for the analysis of the estimator of d which is given in the next section.

2. Estimating the Delay

Given a sample $\{x_t,y_t:\ t = 0,\ldots,T-1\}$, compute the discrete Fourier transforms

$$d_x(\omega) = \Sigma_{t=0}^{T-1} x_t e^{-i\omega t} \tag{15}$$

$$d_y(\omega) = \Sigma_{t=0}^{T-1} y_t e^{-i\omega t} \tag{16}$$

for the discrete frequencies $\frac{2\pi}{T}, \frac{4\pi}{T}, \frac{6\pi}{T}, \ldots, \pi$. Let $\kappa_k = \frac{4\pi k}{T}$ and $\eta_j = \frac{4\pi j}{T}$ denote the discrete frequency values for $\kappa = \omega_1 + \omega_2$ and $\eta = \omega_1 - \omega_2$. From expression (4.3.15) in Brillinger [1], it follows that

$$ET^{-1}d_x(\omega_1)d_x(\omega_2)d_y^*(\omega_1 + \omega_2) = S_{xy}(\omega_1,\omega_2) + O(T^{-1}) \tag{17}$$

where the star denotes taking the complex conjugate, i.e. $d_y^*(\omega) = d_y(-\omega)$. Also from expression (4.3.15), or from Rosenblatt and Van Ness [6],

$$\text{Var}\ T^{-1}d_x(\omega_1)d_x(\omega_2)d_y^*(\omega_1 + \omega_2) = TS_x(\omega_1)S_x(\omega_2)S_y(\omega_1 + \omega_2) + O(T^{-1}) \tag{18}$$

in the interior of the principal domain. Moreover, $d_x(\omega_1)d_x(\omega_2)d_y^*(\omega_1 + \omega_2)$ and $d_x(\omega_3)d_x(\omega_4)d_y^*(\omega_3 + \omega_4)$ are asymptotically independent as $T \to \infty$ if $\omega_1 \neq \omega_3$ or $\omega_2 \neq \omega_4$.

In order to simplify the asymptotics, assume that S_x is a constant for ω in the band $0<\omega<2\pi f_o$ for some f_o. Since the spectrum can be consistently estimated from the sample as $T \to \infty$ using one of several related estimators (see the review by Hinich and Clay [2]), the value of the spectrum in this band can be assumed to be known and equal to one.

Let f_1 be a frequency such that $f_1 < 1 - f_o$, and define T_1 to be the largest integer such that $T_1/T \leq f_1$. As $T \to \infty$, $T_1/T \to f_1$. Consider the smoothed cross bispectrum estimator defined by the expression

$$B(\kappa) = T^{-2}\Sigma_{j=1}^{T_1} d_x[(\kappa+\eta_j)/2]d_x[(\kappa-\eta_j)/2]d_y^*(\kappa). \tag{19}$$

The sum in (19) keeps η_j in the principal domain for each $\kappa<2\pi f_o$, since $\kappa+\eta_j<2\pi f_o + 2\pi T_1/T\leq 2\pi$. Since

$$T^{-1}\Sigma_{j=1}^{T_1}\cos(\eta_j d/2) = \int_0^{f_1} \cos(2\pi xd)\,dx + O(T^{-1}),$$

it follows from (14), (17), and (18) that

$$B(\kappa_k) = (2\pi d)^{-1}A(\sin 2\pi f_1 d)e^{-i(\kappa_k d/2)} + u_k, \tag{20}$$

By the central limit theorem, the error term u_k is approximately complex normal $N(0, f_1 S_y(\kappa_k))$ for large T. Moreover, u_k and $u_{k'}$ are approximately independent if $k \neq k'$.

Equation (20) shows that $B(\kappa_k)$ contains a "hidden periodicity" in k with period T/d. There is no aliasing as long as $d < T/2$. This observation suggests the following estimator of the lag d. Let $\hat{d}$ denote the value of δ in the range $0 < \delta < T/2$ which maximizes the periodogram

$$T_o^{-1}\left|\Sigma_{k=1}^{T_o} B(\kappa_k)e^{i(\kappa_k \delta/2)}\right|^2, \tag{21}$$

where T_o is the largest integer such that $T_o/T \leq f_o$. As long as $f_1 \neq d/2$, the periodogram (21) has a peak of order $O(T_o)$ at $\delta=d$ against a background of order $O(1)$, and thus $\hat{d}$ is a consistent estimator of d.

In order to obtain the asymptotic variance of $\hat{d}$, assume that $S_y(\kappa)$ is constant in the band $0<\kappa<2\pi f_o$. Thus the errors u_k in expression (20) are homoskedastic.

If the u_k variates were independent, the asymptotic properties of $\hat{d}$ are given by Walker [7]. His proof holds for this model where the u_k's are asymptotically independent as $T \to \infty$. Thus $T_o^{3/2}(\hat{d}-d)$ is asymptotically normal with mean zero and variance $6f_1(2\pi d)^2 S_y(|A|\sin 2\pi f_1 d)^{-2}$. Since $T_o/T \to f_o$ as $T \to \infty$, it then follows that for large T,

$$\text{Mean Squared Error of } \hat{d} = \frac{6f_1(2\pi d)^2 S_y}{T^3 f_o^3 |A|^2 \sin^2 2\pi f_1 d}. \tag{22}$$

Note that if f_1 is near zero, the asymptotic variance of $\hat{d}$ is $O(f_1^{-1})$. Several different f_1 values should be used in practise in order to avoid the possibility of setting f_1 near d/2.

This theoretical analysis shows that

$$T^{-1}\left|\sum_{k=1}^{T_o} B(\kappa_o)e^{i(\kappa_k \delta/2)}\right|^2$$

for κ_k in the region where $A(\kappa)$ is slowly varying provides a simple and graphical method for testing for the presence of the nonlinear lagged term in the model relating $\{y_t\}$ or $\{x_t\}$. To give a name to this procedure, call this periodogram the Lagstrum.

3. Multiple Delays

Suppose that

$$y_t = \Sigma_{s=o}^{\infty} h_s x_{t-s} + \Sigma_{\ell=1}^{L} \Sigma_{s=o}^{\infty} a_{\ell s} x_{t-s-d_\ell} x_{t-s} + \varepsilon_t, \tag{23}$$

where $d_1,\ldots,d_L$ are unknown delays. Assume that $\Sigma_s a_{\ell s} = 0$ for each ℓ, or $c_x(d_\ell) = 0$ for $\ell = 1,\ldots,L$. From the previous analysis it follows that the cross bispectrum for $\kappa = \omega_1 + \omega_2, \eta = \omega_1 - \omega_2$ is

$$S_{xy}(\kappa,\eta) = S_x(\omega_1)S_x(\omega_2)\Sigma_{\ell=1}^{L} A_\ell(\kappa)\cos(\eta d_\ell/2)e^{-i(\kappa d_\ell/2)} \tag{24}$$

Assume that S_x and each A_ℓ are flat in the band $(0,f_o)$. Then the large sample Lagstrum will have L peaks of the order $O(T_o)$ at $\delta = d_\ell(\ell = 1,\ldots,L)$ against a background of $O(1)$. There will be no aliasing as long as $T > 2d_\ell$ for each ℓ.

Once again suppose that S_y is also flat in the band. As is shown by Hinich and Shaman [3], the $\hat{d}_1,\ldots,\hat{d}_L$ which maximize the Lagstrum are maximum likelihood estimators of the d_ℓ, ignoring the residual correlation between the u_k error terms. The Lagstrum analysis extends to the multiple lag model as long as T is large and $A_\ell(\omega)$ is slowly varying in the band $(0,f_o)$ for each $\ell = 1,\ldots,L$.

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